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Transient anomaly imaging by the acoustic radiation force[☆]

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ABSTRACT

This paper is devoted to provide a solid mathematical foundation for a promising imaging technique based on the acoustic radiation force, which acts as a dipolar source. From the rigorously established asymptotic expansions of near- and far-field measurements of the transient wave induced by the anomaly, we design asymptotic imaging methods leading to a quantitative estimation of physical and geometrical parameters of the anomaly.

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1. Introduction

An interesting approach to assessing elasticity is to use the acoustic radiation force of an ultrasonic focused beam to remotely generate mechanical vibrations in organs [23,31]. The acoustic force is generated by the momentum transfer from the acoustic wave to the medium. The radiation force essentially acts as a dipolar source. A spatio-temporal sequence of the propagation of the induced

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transient wave can be acquired, leading to a quantitative estimation of the viscoelastic parameters of the studied medium in a source-free region [10,11].

The aim of this paper is to provide a solid mathematical foundation for this technique and to design new methods for anomaly detection using the radiation force. These reconstruction procedures are based on rigorously established inner and outer asymptotic expansions of the perturbations of the wavefield that are due to the presence of the anomaly.

To be more precise, suppose that an anomaly D of the form

$$D = \epsilon B + z$$

is present, where ϵ is the (small) diameter of D , B is a reference domain, and z indicates the location of D . A spherical wave

$$U_{\bar{y}}(x, t) := \frac{\delta(t - |x - \bar{y}|)}{4\pi |x - \bar{y}|}$$

is generated by a point source located at \bar{y} far away from z . When this wave hits the anomaly D , it is perturbed. We will derive asymptotic expansions of this perturbation near and far away from the anomaly as ϵ tends to 0. In fact, we will derive asymptotic expansions of the perturbation $u - U_{\bar{y}}$ after the high-frequency component is truncated, where u is the solution to

$$\begin{cases} \partial_t^2 u - \nabla \cdot (\chi(\mathbb{R}^3 \setminus \bar{D}) + k\chi(D)) \nabla u = \delta_{x=\bar{y}} \delta_{t=0} & \text{in } \mathbb{R}^3 \times]0, +\infty[, \\ u(x, t) = 0 & \text{for } x \in \mathbb{R}^3 \text{ and } t \ll 0. \end{cases}$$

For example, after truncation of the high-frequency component of the solution, the derived asymptotic expansion far away from the anomaly shows that when the spherical wave $U_{\bar{y}}$ reaches the anomaly, it is polarized and emits a new wave. The threshold of the truncation is determined by the diameter of the anomaly and is of order $O(\epsilon^{-\alpha})$ for $0 \leq \alpha < 1$.

Derivations of asymptotic expansions in this paper are rigorous. They are based on careful and precise estimates of the dependence with respect to the frequency of the remainders in associated asymptotic formulas for the Helmholtz equation. Using the outer asymptotic expansion, we design a time-reversal imaging technique for locating the anomaly from measurements of the perturbations in the wavefield in the far-field. It turns out that using the far-field measurement we can reconstruct the location and the polarization tensor of the anomaly. However, it is known that it is impossible to separate geometric features such as the volume from the physical parameters using only the polarization tensor. We show that in order to reconstruct the shape and to separate the physical parameters of the anomaly from its volume one should use near-field perturbations of the wavefield. Based on such expansions, we propose an optimization problem for recovering geometric properties as well as the parameters of the anomaly. The connection between our expansions and reconstruction methods for the wave equation in this paper and those for the Helmholtz equation is discussed in some detail.

In connection with this work, we shall mention on one hand the papers [33,4,24] for the derivations of asymptotic formula for the Helmholtz equation in the presence of small volume anomalies and on the other hand, the review paper [8] and the recent book [5] on different algorithms in wave imaging.

The paper is organized as follows. We rigorously derive in Section 2 asymptotic formulas for the Helmholtz equation and estimate the dependence of the remainders in these formulas with respect to the frequency. Based on these estimates, we obtain in Section 3 formulas for the transient wave equation that are valid after truncating the high-frequency components of the fields. These formulas describe the effect of the presence of a small anomaly in both the near- and far-field. We then propose different methods for detecting the physical and geometric parameters of the anomaly. A time-reversal method is proposed to locate the anomaly and find its polarization tensor from far-field measurements while an optimization problem is formulated for reconstructing geometric parameters of the anomaly and its conductivity.

2. Asymptotic expansions for the Helmholtz equation

In this section we rigorously derive asymptotic formulas for the Helmholtz equation and estimate the dependence of the remainders in these formulas with respect to the frequency. For doing so, we rely on a layer potential technique.

2.1. Layer potentials

For $\omega \geq 0$, let

$$\Phi_\omega(x) = -\frac{e^{\sqrt{-1}\omega|x|}}{4\pi|x|}, \quad x \in \mathbb{R}^3, \quad x \neq 0, \quad (1)$$

which is the fundamental solution for the Helmholtz operator $\Delta + \omega^2$. For a bounded Lipschitz domain Ω in \mathbb{R}^3 and $\omega \geq 0$, let $\mathcal{S}_\Omega^\omega$ be the single-layer potential for $\Delta + \omega^2$, that is,

$$\mathcal{S}_\Omega^\omega[\varphi](x) = \int_{\partial\Omega} \Phi_\omega(x-y)\varphi(y) d\sigma(y), \quad x \in \mathbb{R}^3, \quad (2)$$

for $\varphi \in L^2(\partial\Omega)$. When $\omega = 0$, \mathcal{S}_Ω^0 is the single-layer potential for the Laplacian. Note that $u = \mathcal{S}_\Omega^\omega[\varphi]$ satisfies the Helmholtz equation $(\Delta + \omega^2)u = 0$ in Ω and in $\mathbb{R}^3 \setminus \overline{\Omega}$. Moreover, if $\omega > 0$, it satisfies the radiation condition, namely,

$$\left| \frac{\partial u}{\partial r} - \sqrt{-1}\omega u \right| = O(r^{-2}) \quad \text{as } r = |x| \rightarrow +\infty \text{ uniformly in } \frac{x}{|x|}. \quad (3)$$

It is well known that the normal derivative of the single-layer potential on Lipschitz domains obeys the following jump relation

$$\frac{\partial(\mathcal{S}_\Omega^\omega[\varphi])}{\partial\nu} \Big|_{\pm}(x) = \left(\pm \frac{1}{2}I + (\mathcal{K}_\Omega^{-\omega})^* \right) [\varphi](x) \quad \text{a.e. } x \in \partial\Omega, \quad (4)$$

for $\varphi \in L^2(\partial\Omega)$, where $(\mathcal{K}_\Omega^{-\omega})^*$ is the singular integral operator defined by

$$(\mathcal{K}_\Omega^{-\omega})^*[\varphi](x) = \text{p.v.} \int_{\partial\Omega} \frac{\partial\Phi_\omega(x-y)}{\partial\nu(y)} \varphi(y) d\sigma(y).$$

Here and throughout this paper the subscripts \pm denote the limit from outside and inside of $\partial\Omega$.

The operator \mathcal{S}_Ω^0 is bounded from $L^2(\partial\Omega)$ into $H^1(\partial\Omega)$ and invertible in three dimensions [32]. Moreover, one can easily see that there exists $\omega_0 > 0$ such that for $\omega < \omega_0$

$$\|\mathcal{S}_\Omega^\omega[\varphi] - \mathcal{S}_\Omega^0[\varphi]\|_{H^1(\partial\Omega)} \leq C\omega\|\varphi\|_{L^2(\partial\Omega)} \quad (5)$$

for all $\varphi \in L^2(\partial\Omega)$ where C is independent of ω . It is also well known that the singular integral operator $(\mathcal{K}_\Omega^0)^*$ is bounded on $L^2(\partial\Omega)$ (see [7] for example). Similarly to (5), one can see that for there exists $\omega_0 > 0$ such that for $\omega < \omega_0$

$$\|(\mathcal{K}_\Omega^{-\omega})^*[\varphi] - (\mathcal{K}_\Omega^0)^*[\varphi]\|_{L^2(\partial\Omega)} \leq C\omega\|\varphi\|_{L^2(\partial\Omega)}$$

for some constant C independent of ω . In view of (4), it amounts to

$$\left\| \frac{\partial(S_\Omega^\omega[\varphi])}{\partial \nu} \Big|_{\pm} - \frac{\partial(S_\Omega^0[\varphi])}{\partial \nu} \Big|_{\pm} \right\|_{L^2(\partial\Omega)} \leq C\omega \|\varphi\|_{L^2(\partial\Omega)}. \quad (6)$$

2.2. Derivations of the asymptotic expansions

Let D be a smooth anomaly with conductivity $0 < k \neq 1 < +\infty$ inside a background medium with conductivity 1. Suppose that $D = \epsilon B + z$, where B is a domain which plays the role of a reference domain, ϵ denotes the small diameter of D , and z indicates the location of D .

Let \bar{y} be a point in \mathbb{R}^3 such that $|\bar{y} - z| \gg \epsilon$, and let

$$V(x, \omega) := \Phi_\omega(x - \bar{y}) = -\frac{e^{\sqrt{-1}\omega|x-\bar{y}|}}{4\pi|x-\bar{y}|}, \quad (7)$$

so that V satisfies

$$\Delta V + \omega^2 V = \delta_{x=\bar{y}}, \quad (8)$$

together with the radiation condition (3).

Let $v(x, \omega)$ be the solution to

$$\nabla \cdot (\chi(\mathbb{R}^3 \setminus \bar{D}) + k\chi(D)) \nabla v + \omega^2 v = \delta_{x=\bar{y}} \quad (9)$$

satisfying the radiation condition (3). In this section, we derive asymptotic expansion formula for $v - V$ as ϵ tends to 0. An important feature of the asymptotic formula derived in this section is a careful estimate of the dependency of the remainder term on the frequency.

Put $w = v - V$. Then w is a unique solution to

$$\nabla \cdot (\chi(\mathbb{R}^3 \setminus \bar{D}) + k\chi(D)) \nabla w + \omega^2 w = (1 - k) \nabla \cdot \chi(D) \nabla V \quad (10)$$

with the radiation condition. In other words, w is the solution to

$$\begin{cases} \Delta w + \frac{\omega^2}{k} w = \left(1 - \frac{1}{k}\right) \omega^2 V & \text{in } D, \\ \Delta w + \omega^2 w = 0 & \text{in } \mathbb{R}^3 \setminus \bar{D}, \\ w|_+ - w|_- = 0 & \text{on } \partial D, \\ \frac{\partial w}{\partial \nu} \Big|_+ - k \frac{\partial w}{\partial \nu} \Big|_- = (k - 1) \frac{\partial V}{\partial \nu}, \\ w \text{ satisfies the radiation condition.} \end{cases} \quad (11)$$

Therefore, w can be represented as

$$w(x, \omega) = \begin{cases} \left(\frac{1}{k} - 1\right) \omega^2 \int_D \Phi_{\frac{\omega}{\sqrt{k}}}(x - y) V(y) dy + \mathcal{S}_D^{\frac{\omega}{\sqrt{k}}}[\varphi](x), & x \in D, \\ \mathcal{S}_D^\omega[\psi](x), & x \in \mathbb{R}^3 \setminus \bar{D}, \end{cases} \quad (12)$$

where $(\varphi, \psi) \in L^2(\partial D)^2$ is the solution to the integral equation

$$\begin{cases} \mathcal{S}_D^{\frac{\omega}{\sqrt{k}}}[\varphi] - \mathcal{S}_D^\omega[\psi] = \left(1 - \frac{1}{k}\right) \omega^2 \int_D \Phi_{\frac{\omega}{\sqrt{k}}}(\cdot - y) V(y) dy, \\ k \frac{\partial \mathcal{S}_D^{\frac{\omega}{\sqrt{k}}}[\varphi]}{\partial \nu} \Big|_- - \frac{\partial \mathcal{S}_D^\omega[\psi]}{\partial \nu} \Big|_+ = (1 - k) \omega^2 \frac{\partial}{\partial \nu} \int_D \Phi_{\frac{\omega}{\sqrt{k}}}(\cdot - y) V(y) dy + (1 - k) \frac{\partial V}{\partial \nu}, \end{cases} \quad (13)$$

on ∂D . The unique solvability of (13) will be shown in the sequel.

Let

$$\tilde{\varphi}(\tilde{x}) = \varphi(\epsilon \tilde{x} + z), \quad \tilde{x} \in \partial B,$$

and define $\tilde{\psi}$ likewise. Then, after changes of variables, (13) takes the form

$$\begin{cases} \mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\varphi}] - \mathcal{S}_B^{\epsilon\omega}[\tilde{\psi}] = F, \\ k \frac{\partial \mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\varphi}]}{\partial \nu} \Big|_- - \frac{\partial \mathcal{S}_B^{\epsilon\omega}[\tilde{\psi}]}{\partial \nu} \Big|_+ = G, \end{cases} \quad \text{on } \partial B, \quad (14)$$

where

$$F(\tilde{x}, \omega) = \left(1 - \frac{1}{k}\right) \epsilon \omega^2 \int_B \Phi_{\frac{\epsilon\omega}{\sqrt{k}}}(\tilde{x} - \tilde{y}) V(\epsilon \tilde{y} + z) d\tilde{y}, \quad (15)$$

$$G(\tilde{x}, \omega) = (1 - k) \epsilon \omega^2 \frac{\partial}{\partial \nu} \int_B \Phi_{\frac{\epsilon\omega}{\sqrt{k}}}(\tilde{x} - \tilde{y}) V(\epsilon \tilde{y} + z) d\tilde{y} + (1 - k) \frac{\partial V}{\partial \nu}(\epsilon \tilde{x} + z). \quad (16)$$

Define an operator $T : L^2(\partial B) \times L^2(\partial B) \rightarrow H^1(\partial B) \times L^2(\partial B)$ by

$$T(\tilde{\varphi}, \tilde{\psi}) := \left(\mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\varphi}] - \mathcal{S}_B^{\epsilon\omega}[\tilde{\psi}], k \frac{\partial \mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\varphi}]}{\partial \nu} \Big|_- - \frac{\partial \mathcal{S}_B^{\epsilon\omega}[\tilde{\psi}]}{\partial \nu} \Big|_+ \right). \quad (17)$$

We then decompose T as

$$T = T_0 + T_\epsilon, \quad (18)$$

where

$$T_0(\tilde{\varphi}, \tilde{\psi}) := \left(\mathcal{S}_B^0[\tilde{\varphi}] - \mathcal{S}_B^0[\tilde{\psi}], k \frac{\partial \mathcal{S}_B^0[\tilde{\varphi}]}{\partial \nu} \Big|_- - \frac{\partial \mathcal{S}_B^0[\tilde{\psi}]}{\partial \nu} \Big|_+ \right), \quad (19)$$

and $T_\epsilon := T - T_0$. In view of (5) and (6), we have

$$\|T_\epsilon(\tilde{\varphi}, \tilde{\psi})\|_{H^1(\partial B) \times L^2(\partial B)} \leq C \epsilon \omega (\|\tilde{\varphi}\|_{L^2(\partial B)} + \|\tilde{\psi}\|_{L^2(\partial B)}) \quad (20)$$

for some constant C independent of ϵ and ω .

Since $\mathcal{S}_B^0 : L^2(\partial B) \rightarrow H^1(\partial B)$ is invertible, we readily see that $T_0 : L^2(\partial B) \times L^2(\partial B) \rightarrow H^1(\partial B) \times L^2(\partial B)$ is invertible. In fact, we have the following lemma.

Lemma 2.1. For $(f, g) \in H^1(\partial B) \times L^2(\partial B)$ let $(\tilde{\varphi}, \tilde{\psi}) = T_0^{-1}(f, g)$. Then

$$\begin{aligned}\tilde{\varphi} &= \tilde{\psi} + (S_B^0)^{-1}[f], \\ \tilde{\psi} &= \left(\frac{k+1}{2(k-1)}I - (\mathcal{K}_B^0)^* \right)^{-1} \left[\frac{k}{k-1} \left(-\frac{1}{2}I + (\mathcal{K}_B^0)^* \right) (S_B^0)^{-1}[f] - \frac{1}{k-1}g \right].\end{aligned}$$

Thanks to (18) and (20), there is $\epsilon_0 > 0$ such that T is invertible if $\epsilon\omega \leq \epsilon_0$ and

$$T^{-1} = T_0^{-1} + E \quad (21)$$

where the operator E satisfies

$$\|E(f, g)\|_{L^2(\partial B) \times L^2(\partial B)} \leq C\epsilon\omega (\|f\|_{H^1(\partial B)} + \|g\|_{L^2(\partial B)})$$

for some constant C independent of ϵ and ω .

Suppose that $\epsilon\omega \leq \epsilon_0 < 1$. Let $(\tilde{\varphi}_\omega, \tilde{\psi}_\omega)$ be the solution to (14). Then by (21) we have

$$(\tilde{\varphi}_\omega, \tilde{\psi}_\omega) = T_0^{-1}(F, G) + E(F, G). \quad (22)$$

Observe that

$$\|F\|_{H^1(\partial B)} \leq C\epsilon\omega^2. \quad (23)$$

On the other hand, G can be written as

$$G(\tilde{x}) = (1-k)\nabla V(z, \omega) \cdot \nu(\tilde{x}) + G_1(\tilde{x})$$

where G_1 satisfies

$$\|G_1\|_{L^2(\partial B)} \leq C\epsilon\omega^2.$$

Therefore, we have

$$(\tilde{\varphi}_\omega, \tilde{\psi}_\omega) = T_0^{-1}(0, (1-k)\nabla V(z) \cdot \nu) + T_0^{-1}(F, G_1) + E(F, G). \quad (24)$$

Note that

$$\|T_0^{-1}(F, G_1) + E(F, G)\|_{L^2(\partial B) \times L^2(\partial B)} \leq C\epsilon\omega^2.$$

We also need asymptotic expansion of $\frac{\partial \tilde{\varphi}_\omega}{\partial \omega}$ and $\frac{\partial \tilde{\psi}_\omega}{\partial \omega}$. By differentiating both sides of (14) with respect to ω , we obtain

$$\left\{ \begin{array}{l} \mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{k}}} \left[\frac{\partial \tilde{\varphi}_\omega}{\partial \omega} \right] - \mathcal{S}_B^{\epsilon\omega} \left[\frac{\partial \tilde{\psi}_\omega}{\partial \omega} \right] = \frac{\partial F}{\partial \omega} \\ \quad - \frac{\epsilon}{4\pi\sqrt{k}} \int_{\partial B} e^{\sqrt{-1}\frac{\epsilon\omega}{\sqrt{k}}|\cdot-\tilde{y}|} \tilde{\varphi}_\omega(\tilde{y}) d\sigma(\tilde{y}) + \frac{\epsilon}{4\pi} \int_{\partial B} e^{\sqrt{-1}\epsilon\omega|\cdot-\tilde{y}|} \tilde{\psi}_\omega(\tilde{y}) d\sigma(\tilde{y}), \\ k \frac{\partial}{\partial \nu} \mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{k}}} \left[\frac{\partial \tilde{\varphi}_\omega}{\partial \omega} \right] \Big|_- - \frac{\partial}{\partial \nu} \mathcal{S}_B^{\epsilon\omega} \left[\frac{\partial \tilde{\psi}_\omega}{\partial \omega} \right] \Big|_+ = \frac{\partial G}{\partial \omega} \\ \quad - \frac{\epsilon}{4\pi\sqrt{k}} \frac{\partial}{\partial \nu} \int_{\partial B} e^{\sqrt{-1}\frac{\epsilon\omega}{\sqrt{k}}|\cdot-\tilde{y}|} \tilde{\varphi}_\omega(\tilde{y}) d\sigma(\tilde{y}) + \frac{\epsilon}{4\pi} \frac{\partial}{\partial \nu} \int_{\partial B} e^{\sqrt{-1}\epsilon\omega|\cdot-\tilde{y}|} \tilde{\psi}_\omega(\tilde{y}) d\sigma(\tilde{y}) \end{array} \right. \quad (25)$$

on ∂B . One can see from (15) and (16) that

$$\frac{\partial F}{\partial \omega} = O(\epsilon\omega) \quad \text{and} \quad \frac{\partial G_1}{\partial \omega} = O(\epsilon\omega).$$

Using the same argument as before, we then obtain

$$\left(\frac{\partial \tilde{\varphi}_\omega}{\partial \omega}, \frac{\partial \tilde{\psi}_\omega}{\partial \omega} \right) = T_0^{-1} \left(0, (1-k) \nabla \frac{\partial V}{\partial \omega}(z, \omega) \cdot \nu \right) + O(\epsilon\omega), \quad (26)$$

where the equality holds in $L^2(\partial B) \times L^2(\partial B)$.

We obtain the following proposition from Lemma 2.1 (with $f = 0$), (24), and (26).

Proposition 2.2. *Let $(\tilde{\varphi}_\omega, \tilde{\psi}_\omega)$ be the solution to (14). There exists $\epsilon_0 > 0$ such that if $\epsilon\omega < \epsilon_0$, then the following asymptotic expansions hold:*

$$\tilde{\varphi}_\omega = \left(\frac{k+1}{2(k-1)} I - (\mathcal{K}_B^0)^* \right)^{-1} [\nu] \cdot \nabla V(z, \omega) + O(\epsilon\omega^2), \quad (27)$$

$$\tilde{\psi}_\omega = \left(\frac{k+1}{2(k-1)} I - (\mathcal{K}_B^0)^* \right)^{-1} [\nu] \cdot \nabla V(z, \omega) + O(\epsilon\omega^2), \quad (28)$$

and

$$\frac{\partial \tilde{\varphi}_\omega}{\partial \omega} = \left(\frac{k+1}{2(k-1)} I - (\mathcal{K}_B^0)^* \right)^{-1} [\nu] \cdot \nabla \frac{\partial V}{\partial \omega}(z, \omega) + O(\epsilon\omega), \quad (29)$$

$$\frac{\partial \tilde{\psi}_\omega}{\partial \omega} = \left(\frac{k+1}{2(k-1)} I - (\mathcal{K}_B^0)^* \right)^{-1} [\nu] \cdot \nabla \frac{\partial V}{\partial \omega}(z, \omega) + O(\epsilon\omega), \quad (30)$$

where all the equalities hold in $L^2(\partial B)$.

We are now ready to derive the inner expansion of $w = v - V$. Let Ω be a set containing D and let $\tilde{\Omega} = \frac{1}{\epsilon}\Omega - z$. After changes of variables, (12) takes the form

$$w(\epsilon\tilde{x} + z, \omega) = \begin{cases} (\frac{1}{k} - 1)\epsilon^2\omega^2 \int_B \Phi_{\frac{\epsilon\omega}{\sqrt{k}}}(\tilde{x} - \tilde{y}) V(\epsilon\tilde{y} + z) d\tilde{y} + \epsilon S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\varphi}_\omega](\tilde{x}), & \tilde{x} \in B, \\ \epsilon S_B^{\epsilon\omega}[\tilde{\psi}_\omega](\tilde{x}), & \tilde{x} \in \tilde{\Omega} \setminus B. \end{cases} \quad (31)$$

Since

$$\|S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\tilde{\varphi}_\omega] - S_B^0[\tilde{\varphi}_\omega]\|_{H^1(\partial B)} \leq C\epsilon\omega\|\tilde{\varphi}_\omega\|_{L^2(\partial B)},$$

we have

$$w(\epsilon\tilde{x} + z, \omega) = \begin{cases} \epsilon S_B^0[\tilde{\varphi}_\omega](\tilde{x}) + O(\epsilon^2\omega^2), & \tilde{x} \in B, \\ \epsilon S_B^0[\tilde{\psi}_\omega](\tilde{x}) + O(\epsilon^2\omega), & \tilde{x} \in \tilde{\Omega} \setminus B. \end{cases}$$

Here we assumed that $\omega \geq 1$ since the case when $\omega < 1$ is much easier to handle. It then follows from (27) and (28) that

$$w(\epsilon\tilde{x} + z, \omega) = \epsilon S_B^0\left(\frac{k+1}{2(k-1)}I - (\mathcal{K}_B^0)^*\right)^{-1} [v](\tilde{x}) \cdot \nabla V(z, \omega) + O(\epsilon^2\omega^2), \quad \tilde{x} \in \tilde{\Omega}.$$

On the other hand, we have

$$\frac{\partial w}{\partial \omega}(\epsilon\tilde{x} + z, \omega) = \begin{cases} \epsilon S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\frac{\partial \tilde{\varphi}_\omega}{\partial \omega}](\tilde{x}) + O(\epsilon^2\omega), & \tilde{x} \in B, \\ \epsilon S_B^{\frac{\epsilon\omega}{\sqrt{k}}}[\frac{\partial \tilde{\psi}_\omega}{\partial \omega}](\tilde{x}) + O(\epsilon^2\omega), & \tilde{x} \in \tilde{\Omega} \setminus B. \end{cases}$$

Therefore, we have from (29) and (30)

$$\frac{\partial w}{\partial \omega}(\epsilon\tilde{x} + z, \omega) = \epsilon S_B^0\left(\frac{k+1}{2(k-1)}I - (\mathcal{K}_B^0)^*\right)^{-1} [v](\tilde{x}) \cdot \nabla \frac{\partial V}{\partial \omega}(z, \omega) + O(\epsilon^2\omega), \quad \tilde{x} \in \tilde{\Omega}.$$

Let

$$\hat{v}_1(\tilde{x}) := S_B^0\left(\frac{k+1}{2(k-1)}I - (\mathcal{K}_B^0)^*\right)^{-1} [v](\tilde{x}).$$

Note that \hat{v}_1 is a vector-valued function. It is well known that \hat{v}_1 is the solution to

$$\begin{cases} \Delta \hat{v}_1 = 0 & \text{in } \mathbb{R}^3 \setminus \bar{B}, \\ \Delta \hat{v}_1 = 0 & \text{in } B, \\ \hat{v}_1|_- - \hat{v}_1|_+ = 0 & \text{on } \partial B, \\ k \frac{\partial \hat{v}_1}{\partial \nu} \Big|_- - \frac{\partial \hat{v}_1}{\partial \nu} \Big|_+ = (k-1)v & \text{on } \partial B, \\ \hat{v}_1(\tilde{x}) = O(|\tilde{x}|^{-2}) & \text{as } |\tilde{x}| \rightarrow +\infty. \end{cases} \quad (32)$$

We finally obtain the following theorem.

Theorem 2.3. *Let Ω be a bounded domain containing D and let*

$$R(x, \omega) = v(x, \omega) - V(x, \omega) - \epsilon \hat{v}_1 \left(\frac{x - z}{\epsilon} \right) \cdot \nabla V(z, \omega). \quad (33)$$

There exists $\epsilon_0 > 0$ such that if $\epsilon\omega < \epsilon_0$, then

$$R(x, \omega) = O(\epsilon^2 \omega^2), \quad \nabla_x R(x, \omega) = O(\epsilon \omega^2), \quad x \in \Omega. \quad (34)$$

Moreover,

$$\frac{\partial R}{\partial \omega}(x, \omega) = O(\epsilon^2 \omega), \quad \nabla_x \left(\frac{\partial R}{\partial \omega} \right)(x, \omega) = O(\epsilon \omega), \quad x \in \Omega. \quad (35)$$

Note that the estimates for $\nabla_x R$ in (34) and $\nabla_x \left(\frac{\partial R}{\partial \omega} \right)$ in (35) can be derived using (31).

Based on Theorem 2.3 we can easily derive an asymptotic expansion of $v(x, \omega) - V(x, \omega)$ for $|x - z| \geq C > 0$ for some constant C . For doing so, we first define the polarization tensor $M = M(k, B)$ associated with the domain B and the conductivity contrast k , $0 < k \neq 1 < +\infty$, as follows (see [7]):

$$M(k, B) := (k - 1) \int_B \nabla (\hat{v}_1(\tilde{x}) + \tilde{x}) d\tilde{x}. \quad (36)$$

It should be noticed that the polarization tensor M can be explicitly computed for balls and ellipsoids in three-dimensional space. We also list important properties of M [7]:

- (i) M is symmetric.
- (ii) If $k > 1$, then M is positive definite, and it is negative definite if $0 < k < 1$.
- (iii) The following Hashin–Shtrikman bounds

$$\begin{cases} \frac{1}{k-1} \text{trace}(M) \leq \left(2 + \frac{1}{k}\right) |B|, \\ (k-1) \text{trace}(M^{-1}) \leq \frac{2+k}{|B|}, \end{cases} \quad (37)$$

hold [28,13], where trace denotes the trace of a matrix.

It is worth mentioning that the equality in the second inequality in (37) holds if and only if B is an ellipsoid [27].

Note that $u := v - V$ satisfies

$$(\Delta + \omega^2)u = (k - 1) \nabla \cdot \chi(D) \nabla v,$$

with the radiation condition. Therefore, using the Lipmann–Schwinger integral representation

$$v(x, \omega) - V(x, \omega) = (1 - k) \int_D \nabla v(y, \omega) \cdot \nabla \Phi_\omega(x - y) dy,$$

together with the asymptotic expansion of v in D in Theorem 2.3, we obtain that for x away from z , there exists $\epsilon_0 > 0$ such that if $\epsilon\omega < \epsilon_0$, then

$$v(x, \omega) - V(x, \omega) = (1 - k) \int_D \left(\nabla V(y, \omega) + \nabla \hat{v}_1 \left(\frac{y - z}{\epsilon} \right) \cdot \nabla V(z, \omega) \right) \cdot \nabla \Phi_\omega(x - y) dy \\ + O(\epsilon^4 \omega^3).$$

Now if we approximate $\nabla V(y, \omega)$ and $\nabla \Phi_\omega(x - y)$ for $y \in D$ by $\nabla V(z, \omega)$ and $\nabla \Phi_\omega(x - z)$, respectively, we obtain the following theorem.

Theorem 2.4. *Let Ω' be a compact region away from D ($\text{dist}(\Omega', D) \geq C > 0$ for some constant C) and let*

$$R(x, \omega) = v(x, \omega) - V(x, \omega) + \epsilon^3 \nabla V(z, \omega) M(k, B) \nabla \Phi_\omega(x - z). \quad (38)$$

There exists $\epsilon_0 > 0$ such that if $\epsilon \omega < \epsilon_0$, then

$$R(x, \omega) = O(\epsilon^4 \omega^3), \quad x \in \Omega'. \quad (39)$$

Moreover,

$$\frac{\partial R}{\partial \omega}(x, \omega) = O(\epsilon^4 \omega^2), \quad x \in \Omega'. \quad (40)$$

Note that, in view of the asymptotic formulas derived in [25] for the case of a circular anomaly, the range of frequencies for which formula (38) is valid is optimal.

3. Far- and near-field asymptotic formulas for the transient wave equation

Let D be a smooth anomaly with conductivity $0 < k \neq 1 < +\infty$ inside a background medium with conductivity 1. Suppose that $D = \epsilon B + z$ as before.

Let \bar{y} be a point in \mathbb{R}^3 such that $|\bar{y} - z| \geq C > 0$ for some constant C . Define

$$U_{\bar{y}}(x, t) := \frac{\delta(t - |x - \bar{y}|)}{4\pi|x - \bar{y}|}, \quad (41)$$

where δ is the Dirac mass at 0. $U_{\bar{y}}$ is the Green function associated with the retarded layer potentials and satisfies [18,22]

$$\begin{cases} (\partial_t^2 - \Delta)U_{\bar{y}}(x, t) = \delta_{x=\bar{y}}\delta_{t=0} & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ U_{\bar{y}}(x, t) = 0 & \text{for } x \in \mathbb{R}^3 \text{ and } t \ll 0. \end{cases}$$

For $\rho > 0$, we define the operator P_ρ on tempered distributions by

$$P_\rho[\psi](t) = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} \hat{\psi}(\omega) d\omega, \quad (42)$$

where $\hat{\psi}$ denotes the Fourier transform of ψ . The operator P_ρ truncates the high-frequency component of ψ . Since

$$\hat{U}_{\bar{y}}(x, \omega) = V(x, \omega) := \frac{e^{\sqrt{-1}\omega|x-\bar{y}|}}{4\pi|x-\bar{y}|}$$

using the notation in (7), we have

$$P_\rho[U_{\bar{y}}](x, t) = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} V(x, \omega) d\omega = \frac{\psi_\rho(t - |x - \bar{y}|)}{4\pi|x - \bar{y}|} \quad \text{for } x \neq \bar{y}, \quad (43)$$

where

$$\psi_\rho(t) := \frac{2 \sin \rho t}{t} = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} d\omega. \quad (44)$$

One can easily show that $P_\rho[U_{\bar{y}}]$ satisfies

$$(\partial_t^2 - \Delta)P_\rho[U_{\bar{y}}](x, t) = \delta_{x=\bar{y}}\psi_\rho(t) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}. \quad (45)$$

We consider the wave equation in the whole three-dimensional space with appropriate initial conditions:

$$\begin{cases} \partial_t^2 u - \nabla \cdot (\chi(\mathbb{R}^3 \setminus \bar{D}) + k\chi(D))\nabla u = \delta_{x=\bar{y}}\delta_{t=0} & \text{in } \mathbb{R}^3 \times]0, +\infty[, \\ u(x, t) = 0 & \text{for } x \in \mathbb{R}^3 \text{ and } t \leq 0. \end{cases} \quad (46)$$

The purpose of this section is to derive asymptotic expansions for $P_\rho[u - U_{\bar{y}}](x, t)$. For that purpose, we observe that

$$P_\rho[u](x, t) = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} v(x, \omega) d\omega, \quad (47)$$

where v is the solution to (9). Therefore, according to Theorem 2.3, we have

$$P_\rho[u - U_{\bar{y}}](x, t) - \epsilon \hat{v}_1\left(\frac{x - z}{\epsilon}\right) \cdot \nabla P_\rho[U_{\bar{y}}](x, t) = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} R(x, \omega) d\omega.$$

Suppose that $|t| \geq c_0$ for some positive number c_0 (c_0 is of order the distance between \bar{y} and z). Then, we have by an integration by parts

$$\begin{aligned} \left| \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} R(x, \omega) d\omega \right| &= \left| \frac{1}{t} \int_{|\omega| \leq \rho} \frac{d}{d\omega} e^{-\sqrt{-1}\omega t} R(x, \omega) d\omega \right| \\ &\leq \frac{1}{|t|} (|R(x, \rho)| + |R(x, -\rho)|) + \int_{|\omega| \leq \rho} \left| \frac{\partial}{\partial \omega} R(x, \omega) \right| d\omega \\ &\leq C\epsilon^2 \rho^2. \end{aligned}$$

Since $\epsilon \hat{v}_1(\frac{x-z}{\epsilon}) \cdot \nabla P_\rho[U_{\bar{y}}] = O(\epsilon\rho)$, we arrive at the following theorem.

Theorem 3.1. Suppose that $\rho = O(\epsilon^{-\alpha})$ for some $\alpha < 1$. Then

$$P_\rho[u - U_{\bar{y}}](x, t) = \epsilon \hat{v}_1\left(\frac{x - z}{\epsilon}\right) \cdot \nabla P_\rho[U_{\bar{y}}](x, t) + O(\epsilon^{2(1-\alpha)}).$$

We now derive a far-field asymptotic expansion for $P_\rho[u - U_{\bar{y}}]$. Define

$$U_z(x, t) := \frac{\delta(t - |x - z|)}{4\pi|x - z|}. \quad (48)$$

We have

$$P_\rho[U_z](x, t) = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} \Phi_\omega(x - z) d\omega.$$

From Theorem 2.4, we compute

$$\begin{aligned} \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} (v(x, \omega) - V(x, \omega)) d\omega &= -\epsilon^3 \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} \nabla V(z, \omega) M(k, B) \nabla \Phi_\omega(x - z) d\omega \\ &\quad + \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} R(x, \omega) d\omega, \end{aligned}$$

where the remainder is estimated by

$$\int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} R(x, \omega) d\omega = O(\epsilon^{4(1-\frac{3}{4}\alpha)}).$$

Since

$$\begin{aligned} &\int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} \nabla V(z, \omega) M(k, B) \nabla \Phi_\omega(x - z) d\omega \\ &= \int_{\mathbb{R}} \nabla \left(\frac{\psi_\rho(t - \tau - |x - z|)}{4\pi|x - z|} \right) M(k, B) \nabla \left(\frac{\psi_\rho(\tau - |z - \bar{y}|)}{4\pi|z - \bar{y}|} \right) d\tau, \end{aligned}$$

and

$$\epsilon^3 \int_{\mathbb{R}} \nabla P_\rho[U_z](x, t - \tau) \cdot M(k, B) \nabla P_\rho[U_{\bar{y}}](z, \tau) d\tau = O(\epsilon^3 \rho^2),$$

the following theorem holds.

Theorem 3.2. Suppose that $\rho = O(\epsilon^{-\alpha})$ for some $\alpha < 1$. Then the following far-field expansion holds

$$P_\rho[u - U_{\bar{y}}](x, t) = -\epsilon^3 \int_{\mathbb{R}} \nabla P_\rho[U_z](x, t - \tau) \cdot M(k, B) \nabla P_\rho[U_{\bar{y}}](z, \tau) d\tau + O(\epsilon^{4(1-\frac{3}{4}\alpha)})$$

for x away from z .

It should be noted that Theorem 3.2 says that the perturbation due to the anomaly is (approximately) a wave emitted from the point z at $T := |z - \bar{y}|$. The anomaly behaves then like a dipolar source. This is the key point of our approach for designing time-reversal imaging procedure in the next section. We also emphasize that the approximation holds after truncation of the frequencies higher than $\epsilon^{-\alpha}$ ($\alpha < 1$). This has an important meaning in relation to the resolution limit in imaging as explained in the next section. Moreover, from the optimality of the range of frequencies for which formula (38) is valid, it follows that $\alpha < 1$ is indeed the optimal exponent.

4. Reconstruction methods

A model problem for the acoustic radiation force imaging is (46), where \bar{y} is the location of the pushing ultrasonic beam. The transient wave $u(x, t)$ is the induced wave. The inverse problem is to reconstruct the shape and the conductivity of the small anomaly D from either far-field or near-field measurements of u .

Let $w(x, t) := u(x, t) - U_{\bar{y}}(x, t)$. We first present a method for detecting the location z of the anomaly from measurements of w for x away from z . To detect the anomaly one can use a time-reversal technique. The main idea of time-reversal is to take advantage of the reversibility of the wave equation in a non-dissipative unknown medium in order to back-propagate signals to the sources that emitted them. See [19,14,30,20]. Some interesting mathematical works started to investigate different aspects of time-reversal phenomena: see, for instance, [9] for time-reversal in the time-domain, [17, 29,26,15,16] for time-reversal in the frequency domain, and [21,12] for time-reversal in random media.

In the context of anomaly detection, one measures the perturbation of the wave on a closed surface surrounding the anomaly, and retransmits it through the background medium in a time-reversed chronology. Then the perturbation will travel back to the location of the anomaly.

Suppose that we are able to measure the perturbation w and its normal derivative at any point x on a sphere S englobing the anomaly D . The time-reversal operation is described by the transform $t \mapsto t_0 - t$. Both the perturbation w and its normal derivative on S are time-reversed and emitted from S . Then a time-reversed perturbation, denoted by w_{tr} , propagates inside the volume Ω surrounded by S . Taking into account the definition (48) of the outgoing fundamental solution, spatial reciprocity and time-reversal invariance of the wave equation, the time-reversed perturbation w_{tr} due to the anomaly D in Ω should be defined by

$$w_{\text{tr}}(x, t) = \int_{\mathbb{R}} \int_S \left[U_x(x', t-s) \frac{\partial w}{\partial \nu}(x', t_0-s) - \frac{\partial U_x}{\partial \nu}(x', t-s) w(x', t_0-s) \right] d\sigma(x') ds,$$

where

$$U_x(x', t-s) = \frac{\delta(t-s-|x-x'|)}{4\pi|x-x'|}.$$

However, with the high-frequency component of w truncated, we take the following definition:

$$\begin{aligned} w_{\text{tr}}(x, t) = & \int_{\mathbb{R}} \int_S \left[U_x(x', t-s) \frac{\partial P_{\rho}[u - U_{\bar{y}}]}{\partial \nu}(x', t_0-s) \right. \\ & \left. - \frac{\partial U_x}{\partial \nu}(x', t-s) P_{\rho}[u - U_{\bar{y}}](x', t_0-s) \right] d\sigma(x') ds. \end{aligned} \quad (49)$$

According to Theorem 3.2, we have

$$P_\rho[u - U_{\bar{y}}](x, t) \approx -\epsilon^3 \int_{\mathbb{R}} \nabla P_\rho[U_z](x, t - \tau) \cdot p(z, \tau) d\tau$$

where

$$p(z, \tau) = M(k, B) \nabla P_\rho[U_{\bar{y}}](z, \tau). \quad (50)$$

Therefore it follows that

$$\begin{aligned} w_{\text{tr}}(x, t) &\approx -\epsilon^3 \int_{\mathbb{R}} p(z, \tau) \cdot \int_{\mathbb{R}} \int_S \left[U_x(x', t - s) \frac{\partial \nabla_z P_\rho[U_z]}{\partial \nu}(x', t_0 - s - \tau) \right. \\ &\quad \left. - \frac{\partial U_x}{\partial \nu}(x', t - s) \nabla_z P_\rho[U_z](x', t_0 - s - \tau) \right] d\sigma(x') ds d\tau, \\ &\approx -\epsilon^3 \int_{\mathbb{R}} p(z, \tau) \cdot \nabla_z \int_{\mathbb{R}} \int_S \left[U_x(x', t - s) \frac{\partial P_\rho[U_z]}{\partial \nu}(x', t_0 - s - \tau) \right. \\ &\quad \left. - \frac{\partial U_x}{\partial \nu}(x', t - s) P_\rho[U_z](x', t_0 - s - \tau) \right] d\sigma(x') ds d\tau. \end{aligned}$$

Multiplying the equation

$$(\partial_s^2 - \Delta_{x'}) U_x(x', t - s) = \delta_{s=t} \delta_{x'=x}$$

by $P_\rho[U_z](x', t_0 - \tau - s)$, integrating by parts, and using the equation

$$(\partial_s^2 - \Delta_{x'}) P_\rho[U_z](x', t_0 - \tau - s) = \psi_\rho(s - t_0 + \tau) \delta_{x'=z} \quad \text{in } \mathbb{R}^3 \times \mathbb{R},$$

we have

$$\begin{aligned} &\int_{\mathbb{R}} \int_S \left[U_x(x', t - s) \frac{\partial P_\rho[U_z]}{\partial \nu}(x', t_0 - s - \tau) - \frac{\partial U_x}{\partial \nu}(x', t - s) P_\rho[U_z](x', t_0 - s - \tau) \right] d\sigma(x') ds \\ &= P_\rho[U_z](x, t_0 - \tau - t) - P_\rho[U_z](x, t - t_0 + \tau). \end{aligned} \quad (51)$$

It then follows that

$$w_{\text{tr}}(x, t) \approx -\epsilon^3 \int_{\mathbb{R}} p(z, \tau) \cdot \nabla_z [P_\rho[U_z](x, t_0 - \tau - t) - P_\rho[U_z](x, t - t_0 + \tau)] d\tau. \quad (52)$$

The formula (52) can be interpreted as the superposition of incoming and outgoing waves, centered on the location z of the anomaly. To see it more clearly, let us assume that $p(z, \tau)$ is concentrated at $\tau = T := |z - \bar{y}|$, which is reasonable since $p(z, \tau) = M(k, B) \nabla P_\rho[U_{\bar{y}}](z, \tau)$ peaks at $\tau = T$. Under this assumption, the formula (52) takes the form

$$w_{\text{tr}}(x, t) \approx -\epsilon^3 p \cdot \nabla_z [P_\rho[U_z](x, t_0 - T - t) - P_\rho[U_z](x, t - t_0 + T)], \quad (53)$$

where $p = p(z, T)$. It is clearly sum of incoming and outgoing spherical waves.

Formula (53) has an important physical interpretation. By changing the origin of time, T can be set to 0 without loss of generality. By taking Fourier transform of (52) over the time variable t , we obtain that

$$\hat{w}_{\text{tr}}(x, \omega) \propto \epsilon^3 p \cdot \nabla \left(\frac{\sin(\omega|x-z|)}{|x-z|} \right), \quad (54)$$

where ω is the wavenumber. This shows that the anti-derivative of time-reversal perturbation w_{tr} focuses on the location z of the anomaly with a focal spot size limited to one-half the wavelength which is in agreement with the Rayleigh resolution limit. It should be pointed out that in the frequency domain, (54) is valid only for $\lambda = 2\pi/\omega \gg \epsilon$, ϵ being the characteristic size of the anomaly. In fact, according to Theorem 3.2, it is valid for frequencies less than $O(\epsilon^{-\alpha})$ for $\alpha < 1$.

In the frequency domain, suppose that one measures the perturbation $v - V$ and its normal derivative on a sphere S englobing the anomaly D . To detect the anomaly D one computes

$$\hat{w}(x, \omega) := \int_S \left[\Phi_\omega(x - x') \frac{\overline{\partial(v - V)}}{\partial \nu}(x', \omega) - \overline{(v - V)(x', \omega)} \frac{\partial \Phi_\omega}{\partial \nu}(x - x') \right] d\sigma(x'),$$

in the domain Ω surrounded by S . Observe that $\hat{w}(x, \omega)$ is a solution to the Helmholtz equation: $(\Delta + \omega^2)\hat{w} = 0$ in Ω .

An identity parallel to (51) can be derived in the frequency domain. Indeed, it plays a key role in achieving the resolution limit. Applying Green's theorem to $\Phi_\omega(x - x')$ and $\overline{\Phi_\omega(z - x')}$, we have

$$\int_S \left[\Phi_\omega(x - x') \frac{\overline{\partial \Phi_\omega}}{\partial \nu}(z - x') - \overline{\Phi_\omega(z - x')} \frac{\partial \Phi_\omega}{\partial \nu}(x - x') \right] d\sigma(x') = 2\sqrt{-1} \Im m \Phi_\omega(z - x). \quad (55)$$

In view of (55), we immediately find from the asymptotic expansion in Theorem 2.4 that

$$(v - V)(x, \omega) \propto \epsilon^3 \hat{p} \cdot \nabla \left(\frac{\sin(\omega|x-z|)}{|x-z|} \right), \quad (56)$$

where $\hat{p} = M(k, B)\nabla V(z, \omega)$. The above approximation shows that the anti-derivative of $\hat{w}(x, \omega)$ has a peak at the location z of the anomaly and also proves the Rayleigh resolution limit. Note that (54) is in a good agreement with (56) even though the high-frequency component has been truncated.

It is also worth noticing that a formula similar to (56) can be derived in an inhomogeneous medium Ω surrounded by S . We have

$$\begin{aligned} & \int_S \left[\mathcal{G}(x - x', \omega) \frac{\overline{\partial \mathcal{G}}}{\partial \nu}(x' - z, \omega) - \overline{\mathcal{G}(x' - z, \omega)} \frac{\partial \mathcal{G}}{\partial \nu}(x - x', \omega) \right] d\sigma(x') \\ &= 2\sqrt{-1} \Im m \mathcal{G}(x - z, \omega), \end{aligned} \quad (57)$$

where \mathcal{G} is the Green function in the inhomogeneous medium Ω . Identity (57) shows that the sharper the behavior of $\Im m \mathcal{G}$ at z is, the higher is the resolution. It would be quite interesting to see how the behavior of $\Im m \mathcal{G}$ depends on the heterogeneity of the medium.

Once the location z of the anomaly is found, the polarization tensor associated with the anomaly D can be found using the formula in Theorem 3.2. Since $M(k, D) = \epsilon^3 M(k, B)$, we minimize over symmetric positive matrices $M(k, D)$ the quantity

$$\sum_{l=1}^L \left| P_\rho[u - U_{\bar{y}}](x_l, t) + \int_{\mathbb{R}} \nabla P_\rho[U_z](x_l, t - \tau) M(k, D) \cdot \nabla P_\rho[U_{\bar{y}}](z, \tau) d\tau \right|,$$

for L measurement points x_1, \dots, x_L . It is worth emphasizing that the polarization tensor $M(k, D)$ contains the mixed information of volume $|D|$ and the conductivity k of the anomaly and it is not possible to separate these two information from M .

However, from the near-field measurements, the shape and the conductivity of the anomaly D can be approximately reconstructed. In view of Theorem 3.1, we solve the following minimization problem analogously to [3]. Suppose that the location z of the anomaly $D = z + \epsilon B$ and its characteristic size ϵ are known. Let W be a domain containing D and define the functional

$$L(f, k) = \frac{1}{2\Delta T} \int_{T-\frac{\Delta T}{2}}^{T+\frac{\Delta T}{2}} \left\| P_\rho[u - U_{\bar{y}}](x, t) - \epsilon \hat{v}_1\left(\frac{x-z}{\epsilon}\right) \cdot \nabla P_\rho[U_{\bar{y}}](x, t) \right\|_{L^2(W)}^2 dt + \beta \int_W |\nabla f(x)| dx,$$

where k is the conductivity of D , β is a regularization parameter, f is the binary representation of D , i.e.,

$$f(x) = \begin{cases} 1 & \text{if } x \in D, \\ -1 & \text{if } x \notin D, \end{cases}$$

and \hat{v}_1 is the function corresponding to B as defined in (32). Here it suffices to take ΔT to be of order $O(\frac{\epsilon}{\sqrt{k}})$. We then minimize over binary functions f and constants $0 < k < +\infty$

$$\min_{k, f} L(f, k) \quad (58)$$

subject to (32). We may relax the minimization problem (58) to function of bounded variation. We refer to [3] for the details.

Note that we have to choose a window W that is not so small to preserve some stability and not so big so that we can gain some accuracy. We refer to [2] for a discussion on the critical size of the window W that switches between far-field and near-field reconstructions.

5. Concluding remarks

In this paper, based on careful estimates of the dependence with respect to the frequency of the remainders in asymptotic formulas for the Helmholtz equation, we have rigorously derived the effect of a small conductivity anomaly on transient wave. We have provided near- and far-field asymptotic expansions of the perturbation in the wavefield after truncating its high-frequency component. The threshold of the frequency truncation is of order $\epsilon^{-\alpha}$ ($\alpha < 1$) where ϵ is the diameter of the anomaly. We have also designed a time-reversal imaging technique for locating the anomaly from far-field measurements of the perturbations in the wavefield and reconstructing its polarization tensor. Using a near-field asymptotic formula, we have proposed an optimization problem to reconstruct the shape and to separate the physical parameters of the anomaly from its volume. The connection between our expansions and reconstruction methods for the wave equation in this paper and those for the Helmholtz equation has been discussed. It is expected that the method and the results of this paper can be generalized to dynamic elastic imaging which has important applications in medical imaging [11] as well as in seismology [1]. Some progress in this direction has been made in [6].

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